Formula (3.5) makes possible the separation of the basic component in the approximate representation of quasi-normal trajectories. The solution can be refined by various means. For this it is expedient to use the Galerkin method. It is convenient to select the coordinate functions in the form of polynomials of space coordinates. Efficiency of the Galerkin method is explained by that in zero approximation the shape of oscillations can usually be determined fairly accurately. The unknown weighting coefficients at coordinate functions are in this case small, which makes it possible to linearize in the first approximation the system of transcendental equations that link these.

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DYNAMICS OF A GRAVITATING GASEOUS ELLIPSOID

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Dynamics of adiabatic motions of a gravitating perfect gas of constant density filling a certain ellipsoid is considered in the case when velocities are linear functions of coordinates. It is shown that for an adiabatic exponent $\gamma < 4/3$ the spherically symmetric compression of gas into a point is an unstable process. A reasonable approximation of the oscillating gas motion under strong compression for considerable negative gas energy is indicated. Oscillating mode of expansion of a rotating gaseous ellipsoid in vacuum, which obtains also in the absence of gravitational interaction between gas particles, is determined.

Motions of a continuous medium at velocities that are linear functions of coordinates were the subject of numerous investigations, the earliest of which were the classical works of Dirichlet, Dedekind and Riemann on the forms of equilibrium of a gravitating perfect incompressible fluid (see [1, 2]). Dirichlet [3] had also investigated nonlinear oscillations of a fluid ellipsoid, which are extended to small oscillations in the neighborhood of Maclaurin ellipsoids (see [2, 4]). Similar motions of nongravitating perfect gas were investigated in [5 — 8] (*) in the more general case (gas density is not constant in space), and the motion of dust ellipsoids were investigated in [9, 10]. General problems of motion of continuous medium in which velocities are linear functions of coordinates considered in the book [11], where the motion of an ellipsoid of charged fluid is also dealt with.

The motion of a gravitating gas sphere was considered in [12 — 14] as a model of pulsations of variable Cepheid stars. The spherically symmetric motion of a gravitating gas of variable density, similar to that of nongravitating gas [5], was determined in [15]. The motion of gravitating gas ellipsoids was considered in [16, 17] as the model of formation of galaxies and stars from clouds of initially cold gas. It was noted in [16] that the adiabatic motion of a gravitating gaseous ellipsoid for negative energy E, as well as that of an incompressible fluid ellipsoid investigated by Dirichlet in 1860 (see [2—4]) takes place in an oscillatory mode.

It is shown in the part of this paper that deals with the motion for $E < 0$ that for certain parameter values the oscillating mode can be approximated by a sequence of simpler motions of a gravitating dust ellipsoid. A qualitative description of oscillatory modes of a gravitating gaseous ellipsoid with $E > 0$ (expansion of a rotating gaseous cloud in vacuum) appears in Sect. 2 and for $E < 0$ in Sect. 3 below). The determination of the mode of ellipsoid motion with $E < 0$, based on the analysis of singular points and separatrices (see Sect. 4) of the related dynamic system is presented in Sect. 5 (a similar determination of the oscillatory mode for $E > 0$ appears in the paper cited below). The appendix contains the analysis of a similar problem of two-dimensional hydrodynamics.

1. Statement of the problem. The adiabatic motion of a gravitating perfect gas is determined by equations (see [2])

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p - \rho \operatorname{grad} \Phi, \qquad \frac{d\rho}{dt} = -\rho \operatorname{div} \mathbf{v}, \qquad \frac{d}{dt}\left(\frac{\rho^\gamma}{p}\right) = 0 \qquad (1.1)$$

where $\gamma > 1$ is the adiabatic exponent and Φ is the Newtonian potential generated by the whole mass of gas. Let us consider solutions of Eqs. (1.1) that satisfy the following conditions: Euler's coordinates r_i are linear functions of Lagrangian coordinates a_k

$$r_i = F_{ik}(t) a_k, \qquad i, k = 1, 2, 3 \qquad (1.2)$$

(Hence the velocities $v_i = dr_i / dt$ are linear functions of coordinates r_k.) For $a^2 \equiv a_1^2 + a_2^2 + a_3^2 \leqslant 1$ the gas density ρ and pressure p are determined by formulas

$$\rho = \frac{3M}{4\pi V(F)}, \quad p = \alpha \frac{3M(\gamma - 1)(1 - a^2)}{4\pi V^\gamma(F)}, \quad V(F) = \det \| F_{ik}(t) \| \qquad (1.3)$$

*) See also "The oscillating mode of gas cloud expansion in vacuum" by Bogoiavlenskii, O. I. Preprint, the Landau Inst. Teor. Fisiki, 1975.

where α and M are constants and M is the total mass of gas. We assume that for $a^2 > 1$, $\rho = 0$ and $p = 0$. By conditions (1.3) the gas of constant density fills the ellipsoid which is obtained from a unit sphere $a^2 \leqslant 1$ by transformation $F_{ik}(t)$. The pressure is maximum at the ellipsoid center and zero at its surface.

Denoting the ellipsoid semiaxes by d_1, d_2 and d_3 we obtain for the Newtonian potential Φ at point (x_1, x_2, x_3) inside the ellipsoid a formula of the form [2]

$$\Phi(x_1, x_2, x_3) = \frac{3}{4} GM \int_0^\infty \left(\frac{x_1^2}{d_1^2 + s} + \frac{x_2^2}{d_2^2 + s} + \frac{x_3^2}{d_3^2 + s}\right) \times \tag{1.4}$$
$$[(d_1^2 + s)(d_2^2 + s)(d_3^2 + s)]^{-1/2} ds$$

where G is the gravitational constant. By using formulas (1.2) — (1.4) and the method described in [6, 7] it is possible to show that for the considered motions of gas Eqs. (1.1) are equivalent to the following Lagrangian system determinate in the matrix space:

$$\frac{d^2 F_{ik}(t)}{dt^2} = -\alpha \frac{\partial V^{1-\gamma}(F)}{\partial F_{ik}} + \frac{3}{8} GM \frac{\partial U(F)}{\partial F_{ik}} \tag{1.5}$$

$$U(F) = \int_0^\infty [(d_1^2 + s)(d_2^2 + s)(d_3^2 + s)]^{-1/2} ds$$

The investigation of motions of a gravitating perfect gas, described above is, thus, equivalent to the analysis of the motion of a material point in the nine-dimensional space of matrix F_{ik} in a field whose potential is determined by (1.5). Note that system (1.5) depends on the particular characteristic parameter $\beta = 8\alpha / (3GM)$ which cannot be eliminated by a change of time.

Note. Since d_1^2, d_2^2 and d_3^2 are eigenvalues of matrix $F \cdot F^t$, the integrand in (1.5) is expressed in terms of elements of matrix F_{ik} as follows:

$$(d_1^2 + s)(d_2^2 + s)(d_3^2 + s) = \det(F \cdot F^t) + \frac{1}{2} s [(\mathrm{Tr}(F \cdot F^t))^2 - \mathrm{Tr}(FF^tFF^t)] +$$
$$s^2 \mathrm{Tr}(F \cdot F^t) + s^3$$

where F^t is the transposed matrix and $\mathrm{Tr}(X)$ is the spur of matrix X.

2. The oscillating mode of expansion in vacuum of a spinning gaseous ellipsoid. Let us consider the motion of a considerably expanded gravitating gaseous ellipsoid with total energy $E > 0$ and $\gamma > 1$. Let

$$d_1, d_2, d_3 \gg \frac{GM^2}{K} + \left(\frac{\alpha M}{K}\right)^{1/(3\gamma-3)}, \quad K = \frac{M}{2} \sum_{i,k=1} \left(\frac{dF_{ik}}{dt}\right)^2$$

where K is the kinetic energy of gas. Then according to (1.5) coefficients $F_{ik}(t)$ vary in the first approximation by the linear law

$$F_{ik}(t) = A_{ik} t + B_{ik} \tag{2.1}$$

With a suitable selection of constants A_{ik} and B_{ik} the straight line (2.1) intersects at some $t = t_0$ the surface L: $V(F) = 0$, i.e. the ellipsoid is compressed for $t \to t_0$ into a disk along a certain direction. Since this results in an unbounded increase of pressure which opposes compression, while the velocities of gas and the gravitational forces remain constant, the compression is followed by expansion. This process is represented as the elastic reflection of the straight line (2.1) from surface L at the point of intersection at $t = t_0$. This is followed by another change of coefficients $F_{ik}(t)$ along a straight line of the form (2.1) with new constants A_{ik}^1 and B_{ik}^1. This curve may again intersect surface L, which means a new compression of the ellipsoid, and so on. Thus the change of coefficients $F_{ik}(t)$ along the whole length of the time axis t occurs in the first approximation along broken lines which are elastically reflected from surface L at points of intersection.

Since L is a highly curved surface, it is possible to direct the initial straight line (2.1) so that the number of intersections of the broken lines initiated by it with surface L can be as great (but finite) as desired.

The variation of $F_{ik}(t)$ along any segment of the broken line results in an oscillatory mode of the gas with the ellipsoid volume $(V(F))$ varying between its maximum and minimum. The amplitude of oscillations of the ellipsoid volume and of gas density $\rho(t)$ (1.3) can be arbitrarily large and the time τ of each oscillation for high energy E becomes arbitrarily small: $\tau \sim E^{-1/2}$. The above reasoning is evidently true also for $G = 0$, hence the oscillatory mode also obtains in the absence of gravitational interaction between particles of gas.

From the equation of state $p = \rho RT$ of perfect gas and (1.3) we obtain the expression for the temperature

$$T = \alpha(\gamma - 1)R^{-1}(1 - a^2)V^{1-\gamma}(F) \qquad (2.2)$$

where R is the gas constant Pulsations of the volume of gas are obviously accompanied by fluctuations of its temperature and other physical parameters.

The oscillating mode ceases abruptly when the infinitely extended next following segment of the broken line does not intersect surface L. An unbounded free expansion of gas takes place in that case. The realization of an oscillatory mode depends on the presence of spinning of the gas, since in the absence of the latter (matrix $\|F_{ik}(t)\|$ is diagonal), surface L degenerates into three coordinate planes and the broken line has only three rebounds to which correspond three consecutive compressions and expansions of the ellipsoid along orthogonal axes. The obtained oscillatory mode can be used as the model of motion of a spinning gaseous nebula produced by the explosion of a spinning Supernova star.

3. The oscillatory mode of motion for $E < 0$. If we assume that the gas total energy $E < 0$ and that $\gamma < 4/3$ then, as can be readily shown, the ellipsoid semiaxes d_1, d_2 and $d_3 < C$ at any instant of time. We shall prove that the motion of a gravitating gaseous ellipsoid under strong compression is oscillatory. Assuming that the semiaxes are of the same order of magnitude $d_i \sim d \ll (GM/\alpha)^{1/(4-3\gamma)}$, then for $\gamma < 4/3$ we obtain (see (1.5))

$$GM\frac{\partial U(F)}{\partial F_{ik}} \sim GMd^{-2} \gg \alpha\frac{\partial V^{1-\gamma}(F)}{\partial F_{ik}} \sim \alpha d^{3(1-\gamma)-1} \qquad (3.1)$$

Hence the ellipsoid motion is determined by gravitational forces and is, therefore, approximated by the motion of a gravitating dust ellipsoid. According to [10] a dust ellipsoid subjected to gravitational forces is generally compressed into a disk (i.e. $d_1 \to 0$, $d_2 \to C_2 > 0$ and $d_3 \to C_3 > 0$ and $V(F) = d_1d_2d_3 \to 0$). However an unlimited compression of the ellipsoid into a disk is not possible in the presence of pressure, since the velocities of gas and the gravitational forces under such compression remain finite and the pressure which counteracts compression increases unboundedly. Hence the compression into a disk is followed by expansion (the transition occurs as an elastic reflection of the velocity vector dF_{ik} / dt from surface L) and as the result, the semi-axes d_1, d_2 and d_3 become comparable, the motion is again determined by gravitational forces and is approximated by the motion of the dust ellipsoid. This leads to a new compression of the ellipsoid into a disk (possibly along another direction) which after elastic reflection of the velocity vector from surface L is again followed by expansion, and so on ad infinitum.

The validity of this approximation of the oscillatory mode does not necessarily require a strong compression of the ellipsoid, it is sufficient for $\beta = 8\alpha / (3GM) \ll 1$ (by (2.2) this inequality is satisfied when the initial gas temperature is low).

This approximation of the oscillatory mode is exact for strong compression of the ellipsoid $(E \to -\infty)$ or for $\beta \to 0$.

For such values of parameters the unique solutions of system (1.5) that do not reveal oscillations are solutions with spherically symmetric compression [15]. However such mode is unstable. Hence, when at some time interval the ellipsoid motion is close to spherically symmetric, this mode changes subsequently to an oscillatory one. Such oscillations are also accompanied by fluctuations of temperature T (2.2) and of other physical parameters of gas.

4. Investigation of the dynamic system. (1) We transform by conventional methods the Lagrangian system (1.5) into a Hamiltonian system in phase coordinates $q_i = F_{jk}$, $P_i = F_{jk}^{\cdot}$ $(i = 1, \ldots, 9;\ j,\ k = 1, 2, 3)$. The energy E (a Hamiltonian) in these coordinates is of the form $(n = 9)$

$$E = {}^1/_2\,(P_1^2 + \ldots + P_n^2) + \alpha V^{1-\gamma}(q_i) - {}^3/_8\,GMU(q_i) \tag{4.1}$$

We introduce in the phase space new coordinates

$$p_i = P_i\,(\alpha V^{1-\gamma}(q) + {}^3/_8 GMU(q))^{-1/2}, \quad u = U(q)\,(\beta V^{1-\gamma}(q) + U(q))^{-1} \tag{4.2}$$
$$y_i = q_i Q^{-1/2}, \quad Q = q_1^2 + \ldots + q_n^2, \quad \beta = 8\alpha / (3GM)$$

Coordinates y_i run through the unit sphere S^{n-1}: $y_1^2 + \ldots + y_n^2 = 1$, coordinates p_i run through the whole Euclidean space E^n, and coordinate u runs through the interval $0 < u < 1$. Note that for $\gamma = {}^4/_3$ coordinates u and y_i are dependent; below we assume that $\gamma < {}^4/_3$.

The Lagrangian system (1.5) in terms of coordinates (4.2) and time τ_1

$$\frac{d\tau_1}{dt} = \left(\alpha V^{1-\gamma}(q) + \frac{3}{8} GMU(q)\right)^{1/2} Q^{-1/2} V(y_i)^{-1} \tag{4.3}$$

is of the form

$$p_i^{\cdot} = (1-\gamma)(1-u)\left(-\frac{\partial V}{\partial y_i} - \frac{1}{2}p_i W\right) + u\frac{V(y)}{U(y)}\left(\frac{\partial U}{\partial y_i} - \frac{1}{2}p_i W_1\right) \tag{4.4}$$
$$y_i^{\cdot} = V(y)\,(p_i - y_i\,(p_k y_k))$$
$$u^{\cdot} = u(1-u)\left(\frac{V(y)}{U(y)}W_1 - (1-\gamma)W\right)$$
$$W = \frac{\partial V}{\partial y_k}p_k, \quad W_1 = \frac{\partial U}{\partial y_k}p_k$$

(summation is carried out by the recurrent subscript k).Let us consider system (4.4) for

$E \leqslant 0$ which in coordinates (4.2) is of the form

$$E = {}^3/_8 GM\beta^m V(y)^{(1-\gamma)m} (U(y)/u)^{3(1-\gamma)m} (1-u)^{-m} (P + 1 - 2u) \quad (4.5)$$
$$m = 1/(4-3\gamma), \quad P = {}^1/_2 (p_1^2 + \ldots + p_n^2)$$

It follows from this that the region of $E \leqslant 0$ (or $P + 1 - 2u \leqslant 0$) is a bounded set.

Region S_1 in which system (4.4) is determined in coordinates (4.2) is specified by the conditions $0 < u < 1$, $E \leqslant 0$ and $V(y_i) > 0$ (physical properties of solution correspond according to (1.3) to points of surface $V(q_i) = 0$). We supplement region S_1 by the boundary Γ consisting of four components that are determined by the following conditions: Γ_0: $u = 0$; Γ_1: $u = 1$; Γ_2: $V(y_i) = 0$ and Γ_3: $E = 0$. We denote by S the manifold obtained by this addition of the boundary (we have on S : $0 \leqslant u \leqslant 1$, $E \leqslant 0$ and $V(y_i) \geqslant 0$). System (4.4) obviously extends over the boundary components Γ_0, Γ_1 and Γ_3. Using the simple properties of potential $U(q_i)$ (see [2]) it is possible to show that for $V(y_i) = \det \| Y_{jk} \| \to 0$, expressions

$$(V(y)/U(y))\, \partial U(y)/\partial y_i \to 0,$$

hence we complement these expressions at the boundary component Γ_2 by zero which is their limit value. As the result of this supplementary definition system (4.4) becomes continuously extended over the boundary component Γ_2.

It can be verified that all boundary components Γ and their intersections are invariant submanifolds of the dynamic system (4.4) in S, i.e. a trajectory that begins at some component of boundary Γ remains on it all the time. The system defined in this way on the boundary component Γ_0 $(u = 0)$ is identical to the system which defines the motion of a nongravitating gaseous ellipsoid, while the system defined on the boundary component Γ_1 $(u = 1)$ is identical to the system which defines the motion of a gravitating dust ellipsoid. Thus the dynamic system (4.4) on the manifold S, which describes the motion of a gravitating gaseous ellipsoid, also contains the complete imformation about these two limit forms of motion.

2) All singular points of system (4.4) on the manifold S lie for $E \leqslant 0$ and $\gamma < {}^4/_3$ on the boundary Γ and constitute four sets: L, Φ_+, Φ_- and M.

a) Singular points of L $(u = 1, V(y_i) = 0)$ are intersections of invariant submanifolds Γ_1 $(u = 1)$ and Γ_2 $(V(y_i) = 0)$. These singular points are (for $W \neq 0$) nondegenerate, unstable and have two nonzero eigenvalues

$$\lambda_1 = (1-\gamma)W \quad (\text{variable } u) \quad (4.6)$$
$$\lambda_2 = W \quad (\text{variables } y_i)$$

where the directions of related eigenvectors are indicated in parentheses.

The remaining $2n - 2$ zero eigenvalues relate to directions tangent to manifold L. Since $\gamma > 1$, the eigenvalues λ_1 and λ_2 are of opposite signs, consequently, the singular points of L are saddles.

It is convenient to separate set L into two parts: L_+ $(W \geqslant 0)$ and L_- $(W \leqslant 0)$. Each singular point of L_+ has one incoming separatrix along manifold Γ_2 and one outgoing separatrix along manifold Γ_1, while at point L_- the situation is reversed.

b) The singular points Φ_ε are: $(u = 1,\ p_i = \varepsilon 2^{1/2} y_i,\ \varepsilon = \pm 1$ and $y_i = Y_{jk} = 3^{-1/2} Q_{jk}$, where Q_{jk} is an orthogonal matrix. Calculation of eigenvalues of system (4.4) at singular points Φ_ε shows that these are nondegenerate and unstable. Each point of the three-dimensional set Φ_- has a four-dimensional incoming separatrix formed by diagonal solutions with the spherically symmetric kind of compression. These solutions generalize the exact spherically symmetric solutions. The outgoing separatrix is eleven-dimensional and lies on boundary Γ_1 at zero energy level. Because of this the spherically symmetric compression is unstable. The properties of singular points Φ_+ are identical to those of Φ_- for the opposite direction of time.

c) The degenerate singular points M are: $V(y_i) = 0$, $\partial V / \partial y_i = 0$, and p_i and u are arbitrary. At these singular points matrix $\| Y_{jk} \|$ is twice degenerated.

Thus system (4.4) has no stable singular points for $E \leqslant 0$ and $\gamma < 4/3$, and this is one of the causes of the existence of the oscillatory mode.

3) As noted above, the separatrices of singular points of L_+ and L_- lie on the invariant manifolds Γ_1 and Γ_2. Let us consider system (4.4) on these manifolds.

a) On manifold Γ_1 $(u = 1)$ system (4.4) defines the motion of a gravitating dust ellipsoid. This kind of motion was investigated in [10], where it was shown that along each solution with negative energy E the ellipsoid volume $V(F)$ vanishes twice, i.e. the compression state is followed by compression instead of expansion. The ellipsoid is compressed into a disk in the initial and final states for almost all solutions, i.e. $d_1 = 0$, $d_2 \neq 0$ and $d_3 \neq 0$. In terms of coordinates (4.2) this means that nearly all trajectories of system (4.4) on manifold Γ_1 for $E < 0$ have their beginning and end on the manifold of singular points of L ($V(y_i) = 0$ and $u = 1$), or that a separatrix emanating from nearly every singular point of L_+ reaches some singular point of L.

b) System (4.4) on the manifold Γ_2 $(V(y_i) = 0)$ can be explicitly integrated. Trajectories of this system in time τ defined by

$$d\tau = 2^{1/2} (\gamma - 1)(1 - u) \mid \operatorname{grad} V(y_i^\circ) \mid d\tau_1,$$

are specified by formulas

$$\begin{aligned} &y_i = y_i^\circ, \quad p_i = (2^{1/2} s_i (\sin\tau - \sin\tau_0) + p_i^\circ \cos\tau_0) / \cos\tau \qquad (4.7)\\ &u = \cos^2\tau_0 / \cos^2\tau \\ &s_i = \operatorname{grad} V(y_i^\circ) / \mid \operatorname{grad} V(y_i^\circ) \mid; \quad y_i^\circ,\ \tau_0,\ p_i^\circ = \text{const}, \\ &V(y_i^\circ) = 0 \\ &\sum_{k=1}^{n} p_k^\circ s_k = 2^{1/2} \operatorname{tg} \tau_0 < 0, \quad |\tau_0| < \frac{\pi}{2}, \quad \sum_{k=1}^{n} p_k s_k = 2^{1/2} \operatorname{tg} \tau \end{aligned}$$

The trajectory (4.7) is determinate for $\tau_0 < \tau < -\tau_0$ and runs from the singular point $(p_i^\circ,\ y_i^\circ,\ u = 1)$ belonging to L_- to the singular point $(p_i^1 = p_i(-\tau_0),\ y_i^\circ,\ u = 1)$ belonging to L_+ (consequently all trajectories (4.7) are separatrices of singular points of L_- and L_+). It will be readily seen that the end point of trajectory (4.7) $(p_i^1 = p_i(-\tau_0))$ is obtained from the starting point (p_i°) by its reflection in a plane tangent to surface $V(y_i) = 0$ at point (y_i°).

These results yield the following separatrix pattern:

$$\ldots \xrightarrow{\text{II}} L_+ \xrightarrow{\text{I}} L_- \xrightarrow{\text{II}} L_+ \xrightarrow{\text{I}} \ldots \qquad (4.8)$$

where the transformations indicated by arrows denote the passing from the starting to the end point along the separatrix. Transformations I and II are realized by the separatrices running over manifolds Γ_1 and Γ_2, respectively. Separatrix transformations between the sets L_+, L_- and Φ_ε, M are not shown in the pattern (4.8), because the whole unbounded sequence of transformations (4.8) does not lead out beyond the sets L_+ and L_- for almost all singular points of L_+ and L_-.

5. Derivation of the oscillatory mode of ellipsoid motion for $E < 0$. The unbounded sequence of separatrices determined by the pattern (4.8) is an approximation of trajectories of system (4.4) for considerable negative energy E, and also, for $\beta = 8\alpha / (3GM) \to 0$. In fact function E (see (4.5)) is everywhere bounded below on manifold S, except at the boundary components Γ_1 $(u = 1)$ and Γ_2 $(V(y_i) = 0)$, where $E \to -\infty$. Hence the trajectories of system (4.4) with considerable negative energy E remain at all times in a small neighborhood of manifolds Γ_1 and Γ_2 (this is valid also for any $E < 0$, but then $\beta \to 0$, see (4.2) which defines coordinate u). Consequently these trajectories run along the trajectories of system (4.4) on manifolds Γ_1 and Γ_2, i.e. the general trajectory of system (4.4) runs along the sequence of separatrices of singular points of L_+ and L_-.

The obtained approximation of trajectories of system (4.4) by the sequence of separatrices (4.8) shows that the over-all motion of a gravitating gaseous ellipsoid with considerable negative energy E or small parameter β has a pulsating, oscillating character. According to approximation (4.8) the trajectory appears periodically in the neighborhood of singular points of L_+ and L_- where $\det Y_{jk} = V(y_i) = 0$, i.e. the ellipsoid is periodically compressed into a disk. Moreover it follows from the equation

$$dV(q_i) / d\tau_1 = V(q_i) W \qquad (5.1)$$

that the ellipsoid volume $V(q_i)$ reaches its maximum when the trajectory of system (4.4) runs along the separatrix transformation I, and when the trajectory runs along the separatrix transformation II it reaches its minimum (see (4.8)). The gas density $\rho(t)$ (1.3) in the ellipsoid has also an oscillatory character. By virtue of the relationship

$$dt = d\tau_1 |E|^{-3/2} \beta^{-1} \alpha^l U(y) V(y) u^{-1} |P + 1 - 2u|^{3/2}, \quad l = (2-\gamma)/(1-\gamma)$$

(see (4.3) and (4.5)) the period of each pulsation of the ellipsoid becomes arbitrarily small for $E \to -\infty$.

For $E \to -\infty$ the described pulsating motion of the ellipsoid occurs in a state of considerable compression, since the quantity (see (4.2))

$$d_1^2 + d_2^2 + d_3^2 = Q = \beta^{-2m} (U(y)(1-u)/u)^{2m} V(y)^{2m(\gamma-1)} \qquad (5.2)$$

tends to zero for $E \to -\infty$ (i.e. either $V(y) \to 0$ or $u \to 1$). Note that in the presence of gas spinning the ellipsoid cannot be reduced beyond a definite size. The first integrals J and K (related to the total moment of momentum of gas and to the vortex) of a Lagrangian system of the kind (1.5) are [7]

$$J = F \cdot F^{\cdot t} - F^{\cdot} \cdot F^t, \quad K = F^t \cdot F^{\cdot} - F^{\cdot t} \cdot F \qquad (5.3)$$

It can be shown that for $E \leqslant 0$, $C \neq 0$ and $C = \max\{|J|, |K|\}$ the following inequalities are valid:

$$A^2 B^s / \ln B < d_i < (\beta(\gamma - 1))^{-m}$$

$$A = {}^2/_3\, C\, ((\gamma - 1)\,/\,GM)^{1/3}, \quad B = D\,/\,2\ln D$$
$$D = A\,(\beta\,(\gamma-1))^{m/2}, \quad m = 1\,/\,(4-3\gamma), \quad s = 2\,/\,(\gamma-1)$$

The separatrix approximation (4.8) means that the motion of the trajectory system (4.4) occurs asymptotically for $E \to -\infty$ or $\beta \to 0$ in coordinates y_i as follows.

1) The motion in region $V(y_i) > 0$ takes place along the trajectories corresponding to the gravitation dust ellipsoid. In the general case that trajectory intersects surface $V(y_i) = 0$ at some point y_i° (transformation I).

2) At the point of intersection the trajectory is elastically reflected from surface $V(y_i) = 0$ (transformation II, see (4.8)).

3) After that the motion takes place again along the trajectory corresponding to the gravitating dust ellipsoid up to the next intersection with the surface $V(y_i) = 0$, and so on.

Thus it can be said that the model of the oscillatory mode of motion of the gravitating gaseous ellipsoid is provided by a multidimensional billiard in the region $\det\|Y_{jk}\| = V(y_i) \geqslant 0$ in an eight-dimensional space S^8 $(\mathrm{Tr}\,(Y_0 Y^t) = y_1^2 + \ldots + y_9^2 = 1)$ with an elastically reflecting boundary $\det\|Y_{jk}\| = 0$. Between collisions with the boundary the point moves along trajectories that define the motion of a gravitating dust ellipsoid. The presence of hydrodynamic pressure is revealed by the elastic reflection of a trajectory from the boundary $\det\|Y_{jk}\| = 0$.

Note. Since on manifolds Γ_1 and Γ_2 system (4.4) is independent of γ, there exists for $\gamma > {}^4/_3$ an oscillatory mode ellipsoid motion which is approximated by the separatrix pattern (4.8). The physical content of such oscillatory mode is, however, entirely different. Since by virtue of (4.5) function E' in the neighborhood of manifolds Γ_1 and Γ_2 is for $\gamma > {}^4/_3$ close to zero, and by (5.2) the quantity $d_1^2 + d_2^2 + d_3^2$ is infinitly great, hence an ellipsoid in oscillatory mode has a small energy when $E < 0$ and $\gamma > {}^4/_3$, and the gas is considerably rarefied. For $E \to 0$ the period of each oscillation can be arbitrarily great.

6. Appendix. The problem of spreading of a spinning fluid ellipse in the theory of shallow water (when $\gamma = 2$) is the analog in two-dimensional hydrodynamics of the problem of gaseous ellipsoid expansion in vacuum. That problem was integrated in quadrature in [8]. Let us show that an oscillatory mode, similar to that described in Sect. 2, obtains also in that problem.

We introduce in the space of two-dimensional matrices F_{ij} the coordinates r, φ, φ_1 and φ_2

$$\begin{Vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{Vmatrix} = \begin{Vmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{Vmatrix} \cdot \begin{Vmatrix} r\cos\varphi & 0 \\ 0 & r\sin\varphi \end{Vmatrix} \cdot \begin{Vmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{Vmatrix}$$

In the two-dimensional problem a Lagrangian system similar to (1.5) has the Lagrangian $(\sigma = +1)$

$$L = {}^1/_2\,(r^{\cdot 2} + r^2\varphi^{\cdot 2} + (\varphi_1^{\cdot 2} + \varphi_2^{\cdot 2})\,r^2 + 2\varphi_1^{\cdot}\varphi_2^{\cdot} r^2 \sin 2\varphi) - 2\sigma\,(r^2 \sin 2\varphi)^{-1} \quad (6.1)$$

The lasting momenta $p_{\varphi_1} = \partial L\,/\,\partial\varphi_1^{\cdot} = J$ and $p_{\varphi_2} = \partial L\,/\,\partial\varphi_2^{\cdot} = K$ which coincide with the integrals (5.3) correspond to the cyclic coordinates φ_1 and φ_2.

In the phase coordinates $p_r = \partial L/\partial r^{\cdot}$, $p_\varphi = \partial L\,/\,\partial\varphi^{\cdot}$, r and φ the Lagrangian system with Lagrangian (6.1) becomes a Hamiltonian system with the Hamiltonian

$$H = {}^1/_2 p_r^2 + ({}^1/_2 p_\varphi^2 + U(\varphi))\,/\,r^2 \quad (6.2)$$

$$U(\varphi) = \frac{J^2 + K^2 + 2JK \sin 2\varphi}{2\cos^2 2\varphi} + \frac{2\sigma}{\sin 2\varphi}$$

The substitution of coordinates $r = 1/x$ and time $dt = r^2 d\tau$ splits this system into two Hamiltonian systems with Hamiltonians

$$H = -{}^1/_2 p_r^2 - x^2 H_0, \quad H_0 = {}^1/_2 p_\varphi^2 + U(\varphi) \tag{6.3}$$

It follows from (6.2) that (for $\sigma = +1$ and $J \neq -K$) $U(\varphi) \to \infty$ when $\varphi \to 0, \pi/4$. Hence angle φ oscillates in the potential pit determined by the potential $U(\varphi)$ (shown in Fig. 1 by the solid line). Oscillations of angle φ determine the variation of the ratio of the ellipse semiaxes $d_1 = r\cos\varphi$ and $d_2 = r\sin\varphi$ and provide the image of the general oscillatory mode considered in Sect. 2.

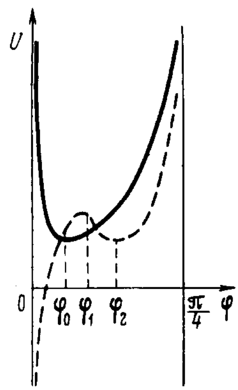

Fig. 1

Let us estimate the number of small oscillations of angle φ in the proximity of the equilibrium state φ_0 ($\partial U / \partial\varphi\,(\varphi_0) = 0$) during the total time T of solution existence, when the x-coordinate varies from zero ($r^2 = d_1^2 + d_2^2 = \infty$) to its maximum $x_* = |H|^{1/2} |H_0|^{-1/2}$ and back to zero. Hence the time $T = \pi (2H_0)^{-1/2}$. The period of small oscillations of angle φ is $T_\varphi \approx 2\pi (U_\varphi''(\varphi_0))^{-1/2}$. Computations show that for $J \approx K \gg 1$ the root of equation $\partial U / \partial\varphi = 0$ is $\varphi_0 \approx (2JK)^{-1/2}$. At that point $H_0 \approx U(\varphi_0) \approx {}^1/_2 (J^2 + K^2)$, $U_\varphi''(\varphi_0) \approx 4(2JK)^{3/2}$

Hence $T = \pi (J^2 + K^2)^{-1/2}$ and $T_\varphi = \pi (2JK)^{-3/4}$. For $J \approx K \to \infty$ the number of oscillations $N = T / T_\varphi$ increases as $J^{1/2}$ and can be arbitrarily high. It can be shown that small oscillations of angle φ with amplitude $\sim \varphi_0^{3/2} \ll \varphi_0$ determine oscillations of the quantity $\sin 2\varphi / 2x^2 = V(F)$.

Let us briefly consider the two-dimensional problem similar to that of compressing a drop of fluid under pressure [18]. The Lagrangian L for that problem is of the form (6.1) with $\sigma = -1$. For $\varphi \to 0$ the potential $U(\varphi) \to -\infty$ and for $\varphi \to \pi/4$, $U(\varphi) \to \infty$ (see the dash line in Fig. 1). For $J \approx -K \gg 1$ and $J \neq -K$ and $0 < \varphi < \pi/4$ potential $U(\varphi)$ has two extrema: maximum φ_1 and minimum φ_2 with $U(\varphi_2) > 0$. Solutions in which angle φ oscillates in the neighborhood of minimum φ_2 have no physical singularities with compression from the rarefied state being followed by expansion. There are apparently no similar solutions of the three-dimensional problem [18].

Note that the infinitely high barrier of potential $U(\varphi)$ for $\varphi = \pi/4$ has a purely geometric origin. In the four-dimensional space of matrices F_{ij} the set of matrices with $d_1 = d_2$ is of second order. Hence for almost all trajectories of the Lagrangian system (6.1) $d_1 \neq d_2$ at all instants of time, and, consequently, nearly all trajectories of the Hamiltonian system (6.3) do not intersect the surface $\varphi = \pi/4$, which is only possible in the presence of an infinite potential barrier. For the same reasons in the three-dimensional problem the relation between the ellipsoid semiaxes $d_1 < d_2 < d_3$ is maintained for almost all motions of gas at all instants of time.

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UDC 534

ON THE PRINCIPAL RESONANCE IN MECHANICS OF CONTINUOUS MEDIA

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Small amplitude resonance oscillations in a nonlinear system close to the stability limit are considered. The Van der Pol equation with a supplementary parameter is derived for the oscillation amplitude; in an autonomous system that parameter defines the dependence of oscillation frequency on amplitude.